THE

AMERICAN

MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII. DECEMBER, 1906. No. 12

ON A CERTAIN CLASS OF CURVES GIVEN BY TRANSCENDENTAL EQUATIONS.

By R. D. CARMICHAEL, Professor of Mathematics, Presbyterian College, Anniston, Alabama.

1. DEFINITION. The equation

(1) $$y^{1/y}=(fx)^{1/fx}$$

is satisfied both by $y=fx$ and by further sets of values of which two examples are $y=4$ or 2, $fx=2$ or 4. Now, $y=fx$ is defined as the *parent* curve, and the other locus is called the *companion* curve.

In a similar manner we define the two curves given by

$(fy)^{1/fy}=(fx)^{1/fx}$, and also by $[f(x,\,y)]^{1/f(x,\,y)}=[F(x,\,y)]^{1/F(x,\,y)}$.

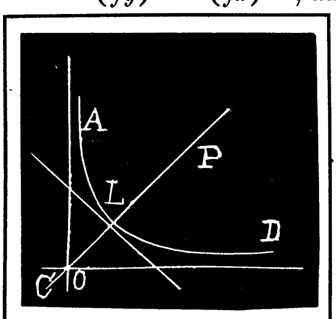


Fig. 1.

EXAMPLES. Figure 1 gives the locus of $y^{1/y}=x^{1/x}$. PC is the straight line $y=x$, and ALD is its companion curve. It is symmetrical with reference to PC. Its asymptotes are $x=1$ and $y=1$.

The locus of $(y^2)^{1/y^2}=(x^2)^{1/x^2}$ has a branch in each quadrant resembling in form ALD above. The figure possesses fourfold symmetry.

As a variation, we plot (Figure 2) the polar companion to the circle, giving the locus of $r^{1/r}=(10\cos\theta)^{1/10\cos\theta}$. The perpendicular to OQ at O is the asymptote in both directions.

2. The Equation of the Companion Curve Alone. If the equation of the parent curve is in the form $y=fx$ both the curve and its companion is given by (1). The values*

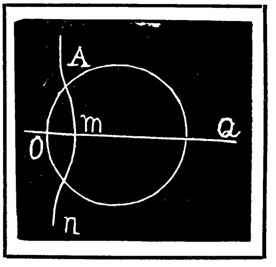
$$y=v^{1/(v-1)} \text{ and } fx=v^{v/(v-1)}, \tag{2}$$

where v is a variable, satisfy equation (1), and define the companion curve alone. It may be shown that equation (1) is satisfied also by

$$y=m^{m/(m-1)} \text{ and } fx=m^{1/(m-1)}. \tag{3}$$

By putting $m=1/v$ and reducing equations (3) we should find equations (2). Hence, when v and m vary through all possible values, the two sets of equations give, in different order, the same set of points.

3. The v-Expressions Imaginary. Since $v^{v/(v-1)}=v.v^{1/(v-1)}$, both of the v-expressions are imaginary or both are real for real values of v. We need therefore attend to only one of them, say, $v^{1/(v-1)}$. If v is any positive integer or fraction or an even negative integer, the expression has at least one real value. But, if it is an odd negative integer, the expression reduces to an even root of a negative quantity, and is hence imaginary. This disposes of all cases except v a negative fraction, say, $-p/q$, p and q positive integers prime to each other. Then



$$v^{1/(v-1)}=\left(-\frac{p}{q}\right)^{-q/(p+q)}=\left(-\frac{q}{p}\right)^{q/(p+q)}. \tag{4}$$

Either p or q is odd or both are odd. If both are odd we have an even root of a negative quantity, which is always imaginary. If one is odd and the other is even, we have an odd root of a real quantity, which always has one real value. Hence,

$v^{1/(v-1)}$ and $v^{v/(v-1)}$ are imaginary if v is a negative odd integer or if it is a negative fraction in its lowest terms and both terms are odd; otherwise they each have at least one real value.

4. Limiting Values of the v-Expressions. The following limiting values are easily determined:

$$v^{1/(v-1)} \doteq \infty\,;\quad v^{v/(v-1)} \doteq 1\,;\quad \text{when } v \doteq 0. \tag{5}$$

$$v^{1/(v-1)} \doteq 1\,;\quad v^{v/(v-1)} \doteq \infty\,;\quad \text{when } v \doteq \infty. \tag{6}$$

$$v^{v/(v-1)} \doteq v^{1/(v-1)} \doteq e\ (=2.718+), \text{ when } v \doteq 1. \tag{7}$$

*Compare Monthly, Vol. XIII, pages 18, 72.

5. THE COMPANION TO THE COMPANION. Since the equation (1) is of the form $Fy = \Psi(x)$, it is evidently included in the corresponding exponential equation

(8) $$(Fy)^{1/Fy} = (\Psi x)^{1/\psi x}.$$

The latter yields the equation of the first companion, and also

(9) $$y^{1/y} = v^{1/(v-1)} \text{ and } (fx)^{1/fx} = v^{v/(v-1)},$$

which are the equations of the companion to the companion.

We propose now to prove that this companion to the companion is in general an imaginary curve.

The minimum value of $v^{1/(v-1)} + v^{v/(v-1)}$ may be shown, by differentiation, to correspond to $v=1$; and therefore by equation (7) it is $2e$. Hence, for any value of v, one or the other of the expressions above is equal to or greater than e. Let it be the first one. Then

(10) $$y^{1/y} \overline{\gtrless} e. \qquad \therefore \frac{\log y}{y} \overline{\gtrless} 1.$$

But the maximum value of $(\log y)/y$ is readily shown to be $1/e$; and therefore the maximum value of $y^{1/y}$ is $e^{1/e}$. Equations (10) can therefore be satisfied only for imaginary values. If we suppose $v^{v/(v-1)} \overline{\gtrless} e$, we should, in the same way, be led to the introduction of imaginaries. Hence,

Companion curves, whose parent equations are expressed by a relation between two variables, themselves have no real companions.

Hereafter, by *companion*, we shall mean the real companion.

6. FORM OF THE EQUATION OF THE PARENT CURVE. The form and even the nature of the companion is often changed by a change in the *form* of the equation of the parent curve. In this way, by a change of origin and axes, an indefinite number of companions may be formed for the same parent curve. A simple illustration will be found by plotting the companions to $y=x$ and $y=a$, a constant.

7. TABLE OF THE v-EXPRESSIONS.

v	$v^{1/(v-1)}$	$v^{v/(v-1)}$	v	$v^{1/(v-1)}$	$v^{v/(v-1)}$
.1	12.910	1.291	2.1	1.963	4.132
.2	7.475	1.495	2.2	1.929	4.244
.3	5.576	1.673	2.3	1.898	4.378
.4	4.605	1.842	2.4	1.869	4.486
.5	4.000	2.000	2.5	1.842	4.605
.6	3.582	2.149	2.6	1.817	4.724
.7	3.279	2.295	2.7	1.793	4.841
.8	3.852	2.442	2.8	1.772	4.962
.9	2.865	2.579	2.9	1.751	5.077

1.0	2.718	2.718	3.0	1.732	5.196
1.1	2.593	2.852	3.5	1.650	5.775
1.2	2.488	2.985	4.0	1.587	6.349
1.3	2.398	3.117	4.5	1.537	6.916
1.4	2.319	3.246	5.0	1.495	7.475
1.5	2.250	3.375	5.5	1.460	8.030
1.6	2.189	3.502	6.0	1.431	8.586
1.7	2.134	3.627	6.5	1.405	9.132
1.8	2.085	3.753	7.0	1.383	9.681
1.9	2.040	3.876	8.0	1.346	10.768
2.0	2.000	4.000	9.0	1.316	11.844

This table will be found of continual value in plotting the companion curves under consideration in the remainder of the paper.

8. THE CURVE $y=x^{1/(x-1)}$. Some investigation of the nature of this locus will serve to aid in our further study. Differentiating, we have

$$\frac{dy}{dx}=x^{1/(x-1)}.\frac{x-1-x\log x}{x(x-1)^2}. \tag{11}$$

For negative values of x the tangent is imaginary; hence, there are no contiguous points in the locus for x negative. But if x and y are both positive it is clear that there is a continuous locus.

In quadrant IV, x is positive and y is negative; hence, for all real points, there is involved an even root of a positive quantity. The points thus determined are real for real values of x. Let $x=p/q$, where the fraction is in its lowest terms, p and q having any positive integer values whatever. This is a general value for x real and positive. Then we have

$$y=\left(\frac{p}{q}\right)^{q/(p-q)}.$$

If y has a negative real value, $p-q$ must be even. Hence, both p and q are odd. Now, consider these values of x,

$$\frac{p}{q}, \quad \frac{(2n+1)p+1}{(2n+1)q}, \quad \frac{(2n+1)p+2}{(2n+1)q},$$

where p and q are both odd and n is a positive integer. For the first and last values there are corresponding real values for y negative; but for the intermediate value y real is positive. If n should be increased at pleasure, remaining always a positive integer, the second and third values of x can be brought indefinitely near the first. But between the first and third values of x, answering to points in the fourth quadrant, there is the intermediate second value above answering to no real point except one in the first quadrant. Therefore, for that part

of the locus which is in the fourth quadrant, x is infinitely discontinuous; that is, the locus has an infinite number of isolated points infinitely near each to another but no two contiguous.

We may in the same manner show that there is an infinitely discontinuous locus in both the second and third quadrant. But it is interesting to note that the discontinuity in the above case arises in a different way from that in these quadrants. In the latter two cases y becomes *imaginary* for the intermediate value of x; but here, besides its imaginary values, it has a real positive value answering to a point in another quadrant.

Confining ourselves now to the real locus we may state the proposition that

The locus of $y=x^{1/(x-1)}$ has a continuous real branch in the first quadrant, and an infinitely discontinuous real branch in each of the other quadrants.

9. COMPANIONS IN SPACE TO PLANE CURVES. If we put z for v in the v-expressions and write the general equation of the companion to the plane curve in the form

$$A \qquad y=z^{1/(z-1)},$$
$$B \qquad fx=z^{z/(z-1)},$$

and look upon A and B as the equations of a curve in space, we may call this the companion in space to the curve $y=fx$. Both equations, as before, may be deduced from (1).

10. COMPANION SURFACES. Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ be the equation of the parent surface, and let the companion be derived from

$$C^{1/C}=D^{1/D}, \text{ where } C\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2},\ D\equiv1-\frac{z^2}{c^2}.$$

The companion alone is

$$\frac{z^2}{c^2}=1-v^{1/(v-1)} \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}=v^{v/(v-1)}.$$

The only values of v which give a continuous locus are positive values, and the only value of $v^{1/(v-1)}$ which can give a real value for z is $v^{1/(v-1)}=1$; for otherwise z is imaginary. For this value, $v^{1/(v-1)}=1$, $z=0$, c being finite, and the entire continuous locus of the companion is the infinite ellipse,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\infty,$$

lying in the plane $z=0$.

The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ has, for one of its companions, the surface

given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=v^{v/(v-1)} \text{ and } \frac{z^2}{c^2}=v^{1/(v-1)}-1.$$

This surface is symmetrical with reference to the plane $z=0$, and incloses that plane by an infinite elliptic boundary. As v increases from 0 to ∞ the sections parallel to $z=0$ decrease continually, remaining ellipses always, until they reach their limiting size, which is that of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Another companion to the same curve is

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=v^{v/(v-1)} \text{ and } \frac{y^2}{b^2}=1-v^{1/(v-1)}.$$

y has a real value only when $v^{1/(v-1)}=1$. Hence the companion degenerates to $y=0$ and $\frac{x^2}{a^2}-\frac{z^2}{c^2}=\infty$. It is therefore an hyperbola infinitely removed from the origin and lying in the plane $y=0$.

The companion to the sphere $x^2+y^2+z^2=a^2$ is

$$x^2+y^2=v^{v/(v-1)} \text{ and } z^2=a^2-v^{1/(v-1)}.$$

When $a^2=1$, the companion is an infinite circle. When $a^2<1$, the companion is imaginary. When $a^2>1$, the companion is a real surface. Every section by planes parallel to $z=0$ is a circle. For all values of v which make $v^{1/(v-1)}>a^2$ the planes which make circular sections are imaginary.

NOTE ON THE ADDITION THEOREM IN TRIGONOMETRY.

By DR. G. A. MILLER.

When $\cos\alpha$, $\cos\beta$, $\sin\alpha$, $\sin\beta$ are substituted for x_1, x_2, y_1, y_2 respectively, in the well known identity

$$(x_1^2+y_1^2)(x_2^2+y_2^2)=(x_1y_2+x_2y_1)^2+(x_1x_2-y_1y_2)^2,$$

there results

$$(\cos^2\alpha+\sin^2\alpha)(\cos^2\beta+\sin^2\beta)=(\cos\alpha\sin\beta+\cos\beta\sin\alpha)^2+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)^2$$

or

$$(\sin\alpha\cos\beta+\cos\alpha\sin\beta)^2+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)^2=1.$$

For every value of and β the two expressions in parenthesis represent real